University of Windsor

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Advanced Hydrology and Water Resources Management

Principles of Groundwater Flow and Groundwater Modeling

Winter 2013

thinking forward

Questions typically asked!

- How much hydraulic head at a point will decline by pumping a nearby well for some specified time?
- What are the expected changes in groundwater levels due to climate change?
- If there is a contaminant spill, where does the plume reach in 5 years, 10 years, etc.?
- What is the capture area for a municipal well?

- How the concentration of a contaminant at a point will change in response to some proposed remedial scheme?

Mathematical framework

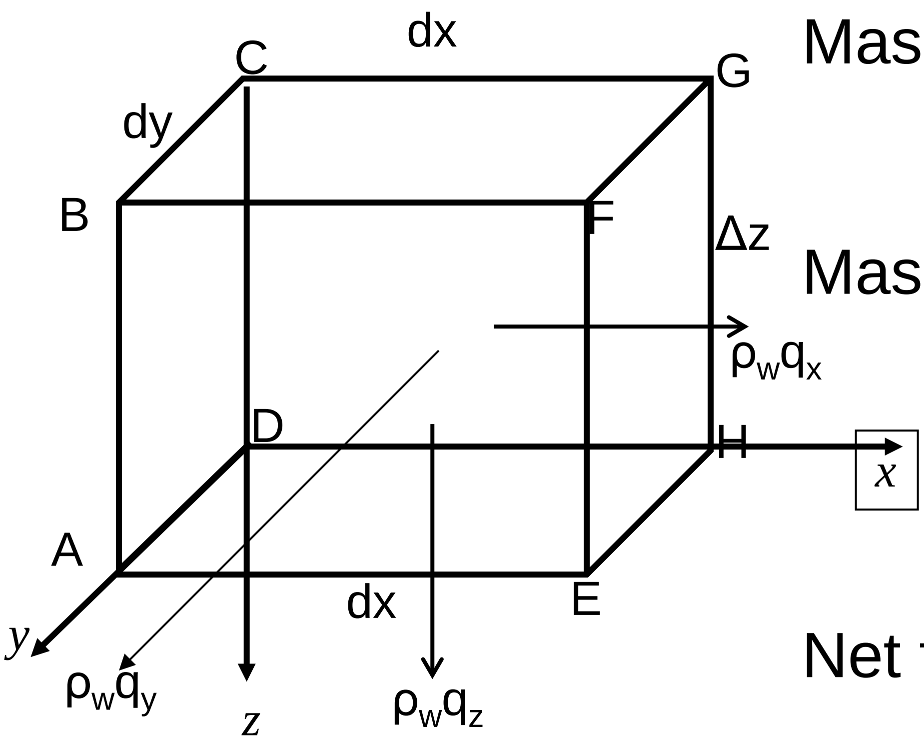
Conceptualization of the problem mathematically,

- Finding the appropriate equations (PDEs) describe the physical phenomena (e.g., flow of groundwater, contaminant transport, etc.)
- Establishing a domain or region where the equation is to be solved
- Defining the conditions along the boundary i.e., boundary conditions
- Solution of the governing equation establishes the hydraulic heads/concentrations at specified (x, y, z) locations

Conservation of Fluid Mass

Mass inflow rate – outflow rate = change in mass storage with time

Any change in mass flowing into the small volume of the aquifer must be balanced by a corresponding change in mass flux out of the volume or a change in the mass stored in the volume



Mass flux into the control volume(CV)

$$= \rho_w q_x \Delta y \Delta z$$

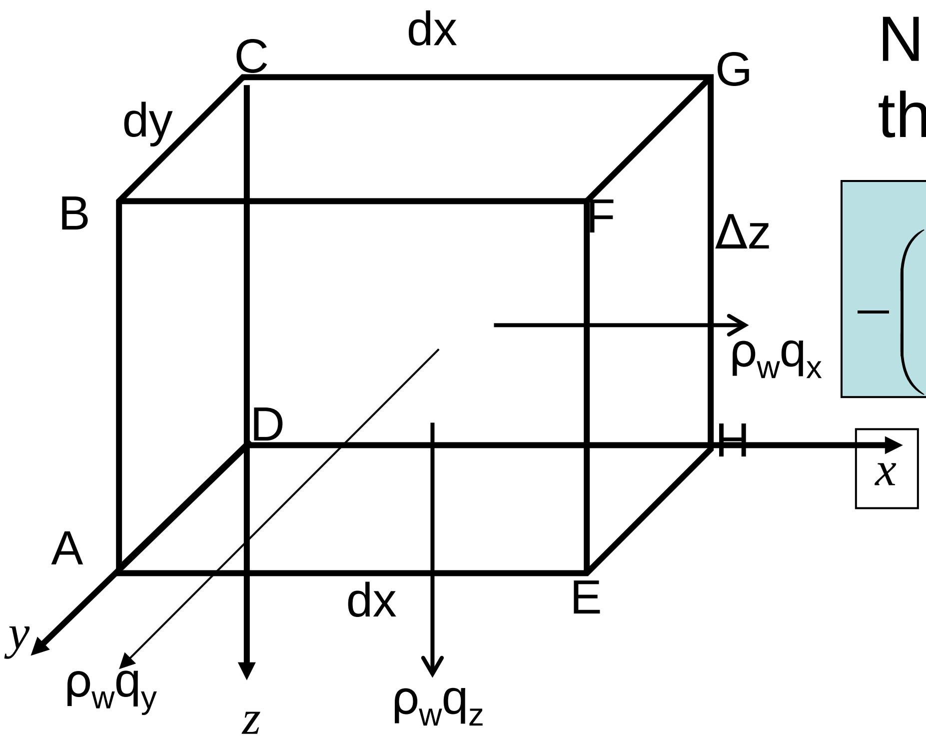
Mass flux out of the CV

$$= \boxed{\left(\rho_w q_x + \frac{\partial(\rho_w q_x)dx}{\partial x}\right)dydz}$$

Net flow rate = $\boxed{-\dfrac{\partial(\rho_w q_x)dxdydz}{\partial x}}$

Conservation of Fluid Mass

Net flow rate may also be determined in y and z directions



Net accumulation of mass in the CV =

$$-\left(\frac{\partial(\rho_w q_x)}{\partial x}+\frac{\partial(\rho_w q_y)}{\partial y}+\frac{\partial(\rho_w q_z)}{\partial z}\right)dxdydz$$

Conservation Fluid Mass

- Volume of water in the CV = ndxdydz
- Initial mass (M) of the water in the CV = $\rho_w$ndxdydz
- Rate of change of mass =

$$\boxed{\frac{\partial M}{\partial t} = \frac{\partial (n\rho_w) dxdydz}{\partial t}}$$

- This can be rewritten as

$$\frac{\partial M}{\partial t} = \left(\alpha \rho_w g + n\beta \rho_w g\right)\rho_w dxdydz \frac{\partial h}{\partial t}$$

To include a quantity that is easier to measure/quantify

Conservation Fluid Mass

- Substituting the expressions for the LHS and RHS terms and dividing both sides by $\Delta x \Delta y \Delta z$

$$-\left(\frac{\partial(\rho_w q_x)}{\partial x}+\frac{\partial(\rho_w q_y)}{\partial y}+\frac{\partial(\rho_w q_z)}{\partial z}\right)dxdydz=\left(\alpha\rho_w g+n\beta\rho_w g\right)\rho_w dxdydz\frac{\partial h}{\partial t}$$

$$-\left(\frac{\partial q_x}{\partial x}+\frac{\partial q_y}{\partial y}+\frac{\partial q_z}{\partial z}\right)=\left(\alpha\rho_w g+n\beta\rho_w g\right)\frac{\partial h}{\partial t}$$

$$-\nabla.(\rho\vec{q})=\frac{\partial(n\rho)}{\partial t}$$

Net fluid outflow rate for the unit volume equals the time rate of change of fluid volume within the unit volume

Equations of Groundwater Flow

- Darcy's law

$$q_x = \left(-K_x \frac{\partial h}{\partial x} \right)$$

$$q_y = \left(-K_y \frac{\partial h}{\partial y} \right)$$

$$q_z = \left(-K_z \frac{\partial h}{\partial z} \right)$$

Isotropic porous media

$$q_x = \left(-K \frac{\partial h}{\partial x} \right)$$

$$q_y = \left(-K \frac{\partial h}{\partial y} \right)$$

$$q_z = \left(-K \frac{\partial h}{\partial z} \right)$$

Substituting for q_x, q_y, and q_z

Conservation Fluid Mass

$$\frac{\partial}{\partial x}\left(K_x \frac{\partial h}{\partial x}\right)+\frac{\partial}{\partial y}\left(K_y \frac{\partial h}{\partial y}\right)+\frac{\partial}{\partial z}\left(K_z \frac{\partial h}{\partial z}\right)=S_s \frac{\partial h}{\partial t}$$

$$K\left(\frac{\partial}{\partial x}\left(\frac{\partial h}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{\partial h}{\partial y}\right)+\frac{\partial}{\partial z}\left(\frac{\partial h}{\partial z}\right)\right)=S_s \frac{\partial h}{\partial t}$$

$$\left(\frac{\partial^2 h}{\partial x}+\frac{\partial^2 h}{\partial y}+\frac{\partial^2 h}{\partial z}\right)=\frac{S_s}{K}\frac{\partial h}{\partial t}$$

Diffusion Equation

(S_s/K) = hydraulic diffusivity

$$\frac{\partial^2 h}{\partial x}+\frac{\partial^2 h}{\partial y}+\frac{\partial^2 h}{\partial z}=0$$

Laplace Equation (Steady state)

$$\nabla^2(\quad)=\frac{\partial^2}{\partial x}+\frac{\partial^2}{\partial y}+\frac{\partial^2}{\partial z} \qquad \nabla^2 h=\frac{S_s}{K}\frac{\partial h}{\partial t}$$

Solution of Governing Equations

Mathematical models (of a physical system for example) must have certain initial or boundary conditions applied in order to solve the problem

<u>Solution</u>

- Solve Equations mathematically
  - Analytical – Exact solution (usually for simple systems, simple geometry ie., 1D/2D and Isotropic, homogeneous)
  - Numerical – allows for complex conditions
  - Analog Models –
    - Electrical Resistivity –Hydraulic processes ≈ electrical
      - Capacitance => S_s, Resistors => 1/K, Volt => h
- Graphical solutions → Flow Nets -> 2D-Steady state
- Interpret mathematical results in terms of physical problem

Boundary and Initial Conditions

- A potential field is presumed to exist i.e., h(x,y,z,t) is well-defined scalar quantity
  - h(x,y,z,t) changes over space and time

- The changes in potential over space results in gradient. This gradient is a vector perpendicular to the equipotential lines, that is, it is colinear with the flow for an isotropic porous medium.

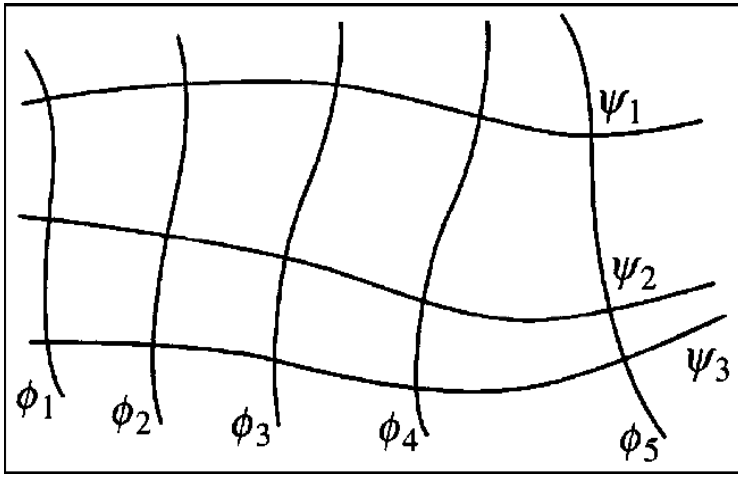
- If divergence (ie., net outflow rate per unit volume) is zero, it is steady state. Else, unsteady state.

- If flow is steady, given the head or the gradient of head on the entire boundary of the region, it is possible to calculate head distribution h(x,y,z).

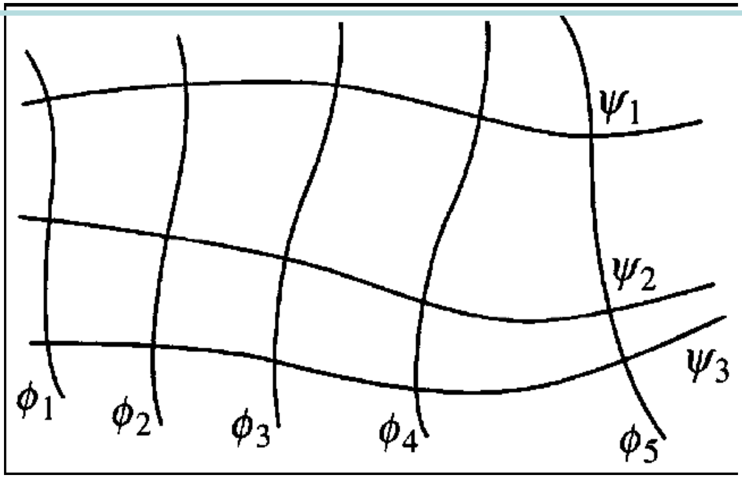
- If flow is unsteady, given the head or the gradient of head on the entire boundary of the region and initial conditions, it is possible to calculate head distribution h(x,y,z,t).

Flow Net

- Flow line – an imaginary line that traces the path that a particle of groundwater would follow as it flows through an aquifer.
- Equipotential line
- In an isotropic aquifer, flow lines and equipotential lines cross at right angles.



Flow Nets



- A network of flow lines and equipotential lines
- <u>A graphical solution approach to solve 2-D steady state equations for homogeneous isotropic media</u>
- In case of anisotropic aquifer, flow lines cross equipotential lines at an angle dictated by the degree of anisotropy
- Assumptions:
  - Aquifer is fully saturated, homogeneous and isotropic.
  - There is no change in potential field with time.
  - Flow is laminar and Darcy's law is valid.
  - The soil and water are incompressible.
  - All boundary conditions are known.

Modeling

Model is

an approximation of the actual system

whose inputs and outputs are measurable hydrological variables, and its structure is a set of equations linking the inputs and outputs

any device that represents an approximation of a field situation

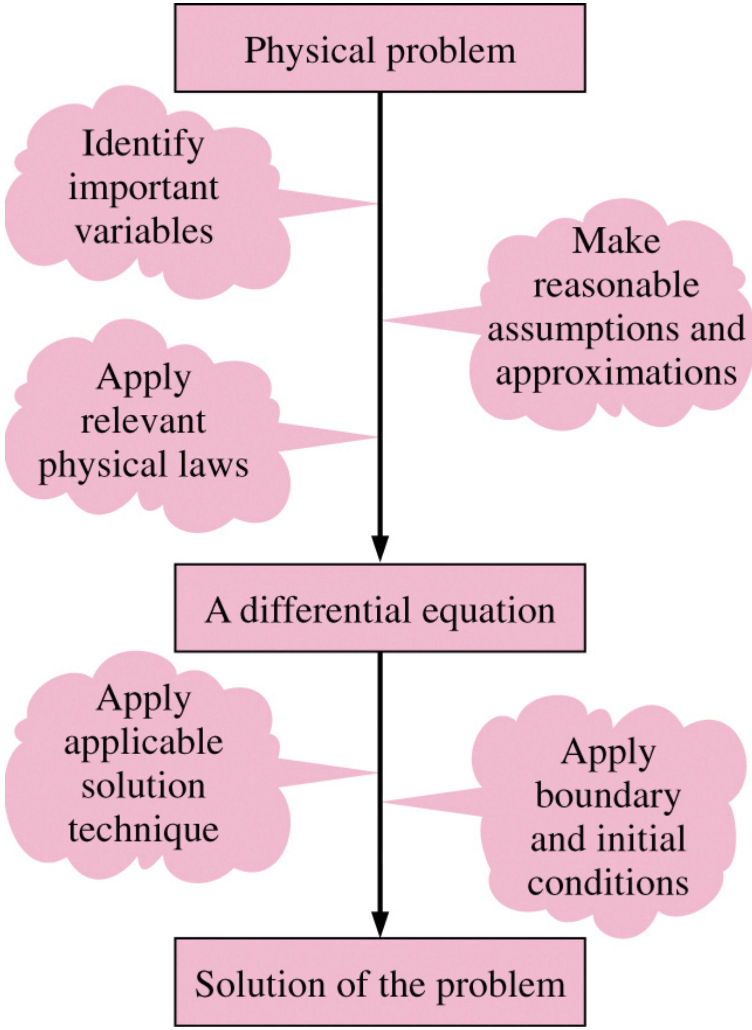
Types of models

- Mathematical – Analytical and Numerical

Mathematical Models

- Simulates groundwater and/or contaminant transport indirectly by means of a governing equation thought to represent the physical processes that occur in the system,
- Together with equations that describe heads or flows along the boundaries of the model.

- The governing equations are solved using numerical techniques such as ***finite difference and finite element*** methods.



Schematic of Modeling framework

(Cengel and Cimbala, 2006)

Steps In Modeling

- Model selection
- Obtain all necessary input data
- Evaluate and refine study objectives in terms of simulations to be performed under various watershed conditions
- Choose methods to determine sub-basin hydrographs and flood routing
- Model Calibration
- Model Verification
- Perform model simulations
- Perform sensitivity analysis
- Evaluate usefulness of the model and comment on needed changes

Conservation Fluid Mass

$$\frac{\partial}{\partial x}\left(K_x\frac{\partial h}{\partial x}\right)+\frac{\partial}{\partial y}\left(K_y\frac{\partial h}{\partial y}\right)+\frac{\partial}{\partial z}\left(K_z\frac{\partial h}{\partial z}\right)=\frac{1}{\rho_w}\frac{\partial(\rho_w n)}{\partial t}$$

$$K\left(\frac{\partial}{\partial x}\left(\frac{\partial h}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{\partial h}{\partial y}\right)+\frac{\partial}{\partial z}\left(\frac{\partial h}{\partial z}\right)\right)=\frac{1}{\rho_w}\frac{\partial(\rho_w n)}{\partial t}$$

$$\left(\frac{\partial^2 h}{\partial x}+\frac{\partial^2 h}{\partial y}+\frac{\partial^2 h}{\partial z}\right)=\frac{S_s}{K}\frac{\partial h}{\partial t} \quad \text{....(30)}$$

Diffusion Equation
(S_s/K) = hydraulic diffusivity

$$\frac{\partial^2 h}{\partial x}+\frac{\partial^2 h}{\partial y}+\frac{\partial^2 h}{\partial z}=0 \quad \text{....(31)}$$

Laplace Equation
(Steady state)

$$\nabla^2(\quad)=\frac{\partial^2}{\partial x}+\frac{\partial^2}{\partial y}+\frac{\partial^2}{\partial z} \qquad \nabla^2 h=\frac{S_s}{K}\frac{\partial h}{\partial t}$$

Boundary Conditions

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Problems in 2-D Space

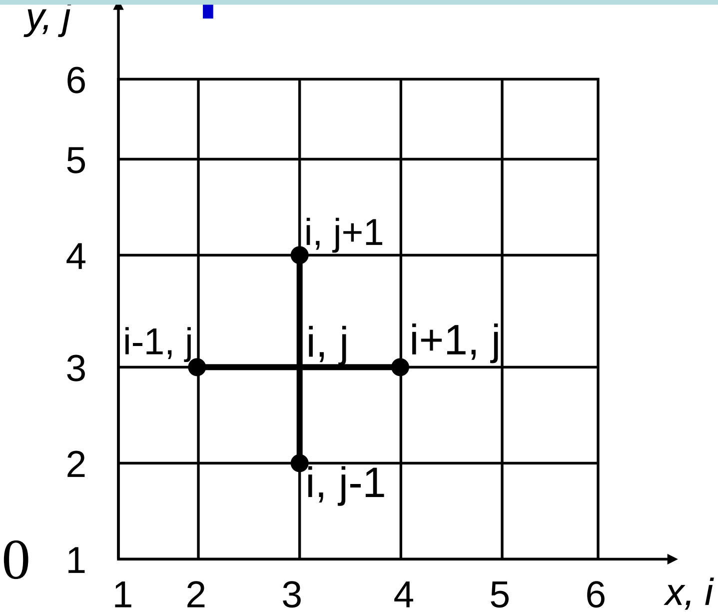
Laplace equation for 2-D steady-state

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$$

Difference equation

$$\left[\frac{\left(u_{i-1,j}-2u_{i,j}+u_{i+1,j}\right)}{\Delta x^2}+\frac{\left(u_{i,j-1}-2u_{i,j}+u_{i,j+1}\right)}{\Delta y^2}\right]=0$$

Rearrange gives:



B.C. defined as constant all around

$$-\Delta y^2 u_{i-1,j}-\Delta x^2 2u_{i,j-1}+2(\Delta x^2+\Delta y^2)u_{i,j}-\Delta x^2 u_{i,j+1}-\Delta y^2 u_{i+1,j}=0$$

Let D = 2 ($\Delta x^2 + \Delta y^2$)

X = - Δx^2

Y = - Δy^2

$$Yu_{i-1,j}+Xu_{i,j-1}+Du_{i,j}+Xu_{i,j+1}+Yu_{i+1,j}=0$$

$$
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\begin{array}{c|c}
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$$\begin{array}{c|c|c} & (2,2) & (3,2) \\ \hline 2,2 & \multirow{2}{*}{$\begin{bmatrix} D & Y \\ Y & D \end{bmatrix}$} & \\ 3,2 & & \end{array} \begin{Bmatrix} u_{2,2} \\ u_{3,2} \end{Bmatrix} = \begin{Bmatrix} -Yu_{1,2} - Xu_{2,1} - Xu_{2,3} \\ - Xu_{3,1} - Xu_{3,3} - Yu_{4,2} \end{Bmatrix}$$

Transient Heat Transfer in 2-D

$$\frac{\partial}{\partial x}\left(K\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(K\frac{\partial u}{\partial y}\right)=S\frac{\partial u}{\partial t}$$

Fully implicit formulation

$$\left[\frac{\left(u_{i-1,j}^{n+1}-2u_{i,j}^{n+1}+u_{i+1,j}^{n+1}\right)}{\Delta x^2}+\frac{\left(u_{i,j-1}^{n+1}-2u_{i,j}^{n+1}+u_{i,j+1}^{n+1}\right)}{\Delta y^2}\right]=\frac{S}{K}\frac{u_{i,j}^{n+1}-u_{i,j-1}^{n}}{\Delta t}$$

$$\Delta y^2u_{i-1,j}^{n+1}+\Delta x^2u_{i,j-1}^{n+1}-\left(2(\Delta x^2+\Delta y^2)+\frac{S}{K}\frac{\Delta x^2\Delta y^2}{\Delta t}\right)u_{i,j}^{n+1}+\Delta x^2u_{i,j+1}^{n+1}+\Delta y^2u_{i+1,j}^{n+1}$$

Flow Term

Storage Term

$$=-\frac{S}{K}\frac{\Delta x^2\Delta y^2}{\Delta t}u_{i,j}^{n}$$

$$Assu\min g\ K_{xy}\left(\frac{\partial^2 u}{\partial x\partial y}\right)=0$$

for this solution

Let D = 2 ($\Delta x^2 + \Delta y^2$)

X = - Δx^2

Y = - Δy^2

$$\rho=\frac{S}{K}\frac{\Delta x^2\Delta y^2}{\Delta t}$$

$$\boxed{Yu_{i-1,j}^{n+1}+Xu_{i,j-1}^{n+1}+\left(D+\rho\right)u_{i,j}^{n+1}+Xu_{i,j+1}^{n+1}+Yu_{i+1,j}^{n+1}=\rho u_{i,j}^{n}}$$

(2,2) (2,3) (2,4) (3,2) (3,3) (3,4) (4,2) (4,3) (4,4) (5,2) (5,3) (5,4)

$$
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Y & 0 & X & D+\rho & X & 0 & Y\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & & & & & \\
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0 & 0 & Y & 0 & X & D & X & 0 & Y\ldots\ldots\ldots\ldots & & & \\
0 & 0 & 0 & Y & 0 & X & D & X & 0 & Y\ldots\ldots\ldots & & \\
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0 & 0 & 0 & 0 & 0 & Y & 0 & X & D & X & 0 & Y \\
0 & 0 & 0 & 0 & 0 & 0 & Y & 0 & X & D & X & 0 \\
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MODFLOW

- Most popular 3-D groundwater flow simulation models
- MODFLOW-2005 is a new version of the finite-difference ground-water model commonly called MODFLOW.
- GWF Process of MODFLOW has been divided into "packages." A package is the part of the program that deals with a single aspect of simulation

- Basic
- Block-Centered Flow
- Layer-Property Flow
- Horizontal Flow Barrier
- Well
- Recharge

•General-Head Boundary
•River
•Drain
•Evapotranspiration
•Strongly Implicit Procedure
•Preconditioned Conjugate Gradient
•Direct Solver

MODFLOW

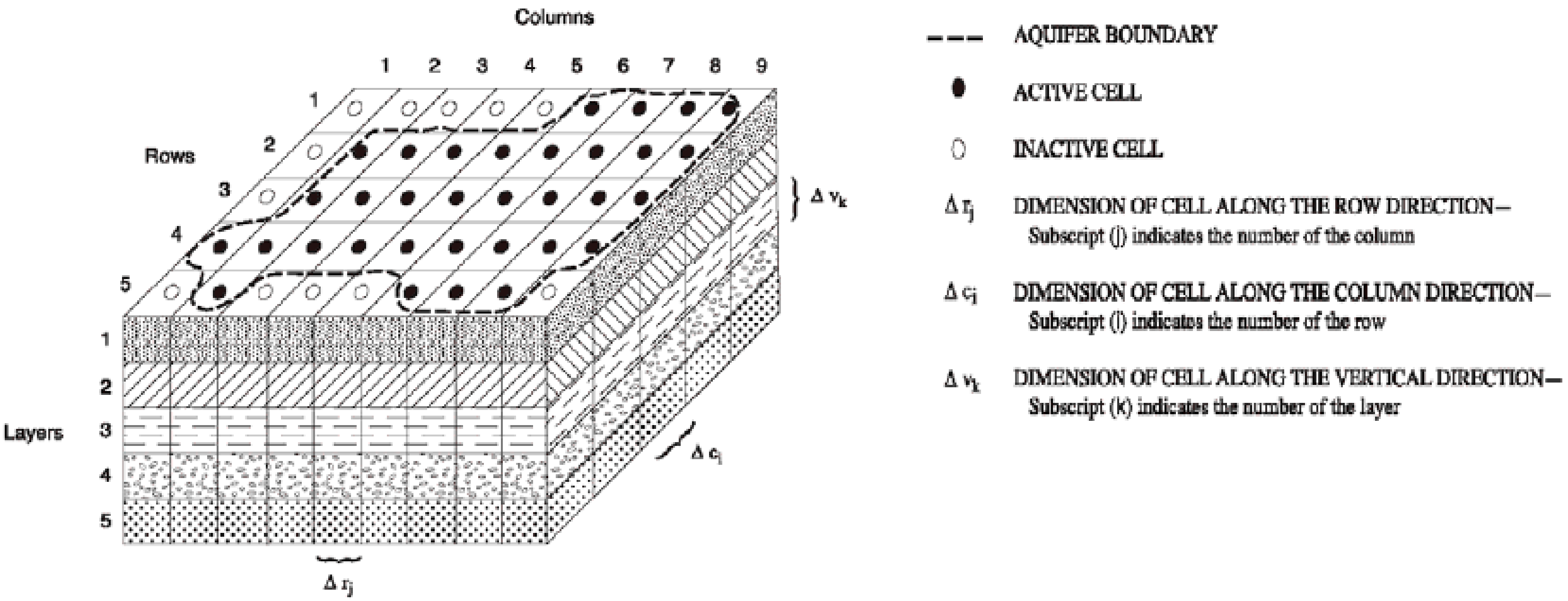
- 3-D movement of ground water of constant density through porous earth material may be described by the partial-differential equation

$$\frac{\partial}{\partial x}\left(K_{xx}\frac{\partial h}{\partial x}\right)+\frac{\partial}{\partial y}\left(K_{yy}\frac{\partial h}{\partial y}\right)+\frac{\partial}{\partial z}\left(K_{zz}\frac{\partial h}{\partial z}\right)+W=S_s\frac{\partial h}{\partial t}$$

- Kxx, Kyy, and Kzz are values of hydraulic conductivity along the x, y, and z coordinate axes, which are assumed to be parallel to the major axes of hydraulic conductivity (L/T);
- h is the potentiometric head (L);
- W is a volumetric flux per unit volume representing sources and/or sinks of water, with W<0.0 for flow out of the ground-water system, and W>0.0 for flow into the system (T-1);
- S_S is the specific storage of the porous material (L-1); and

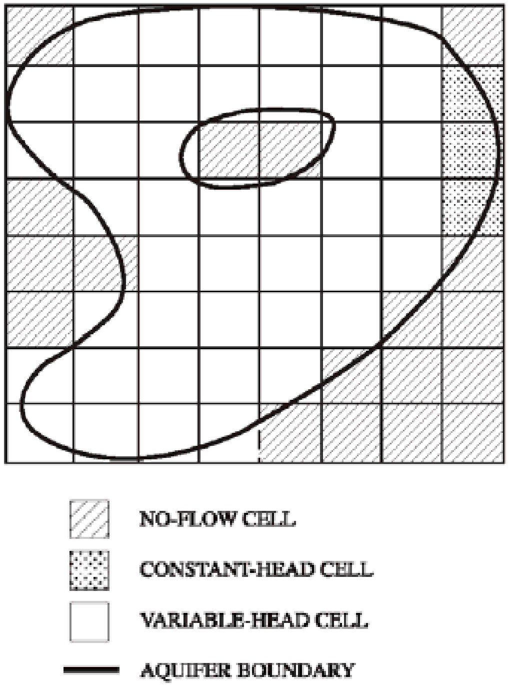
MODFLOW

- simulates steady and nonsteady flow in an irregularly shaped flow system in which aquifer layers can be confined, unconfined, or a combination of confined/unconfined.

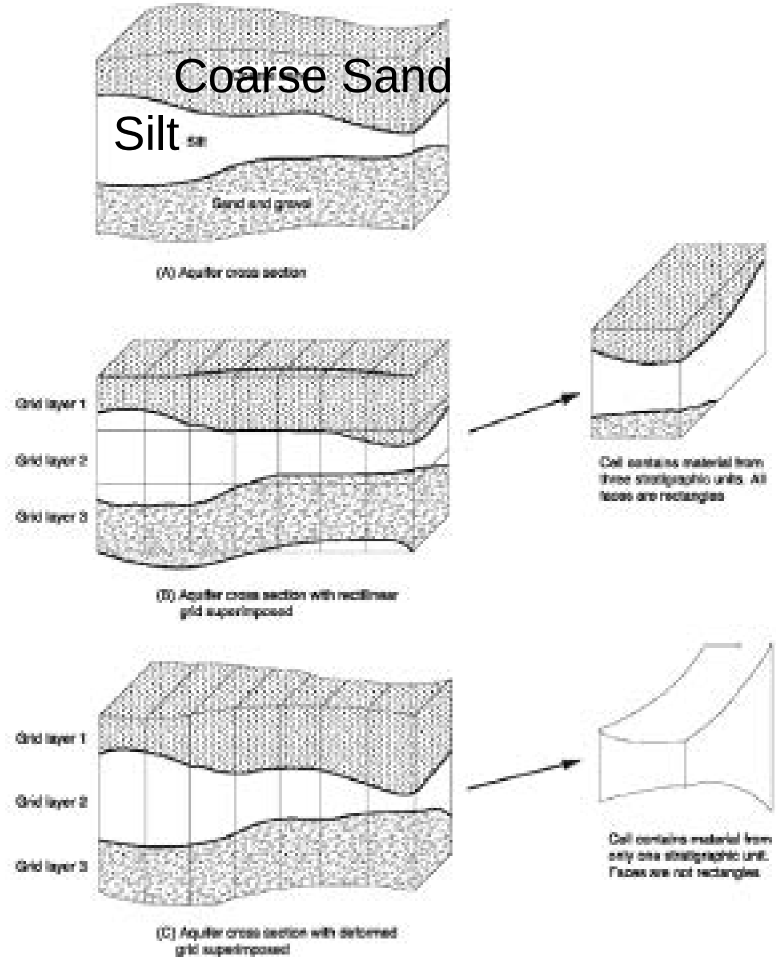


A discretized hypothetical aquifer system. (Modified from McDonald and Harbaugh, 1988.)

MODFLOW

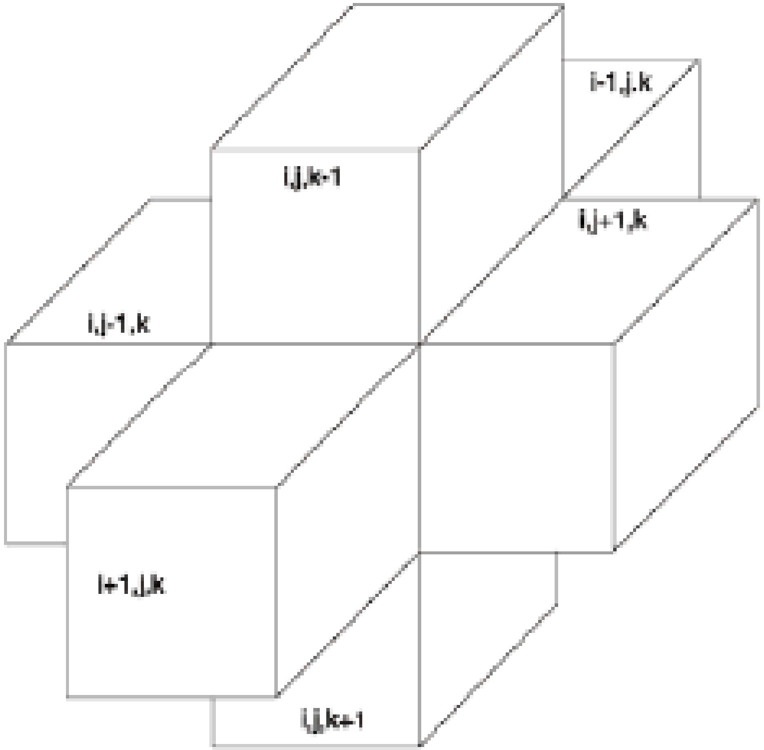


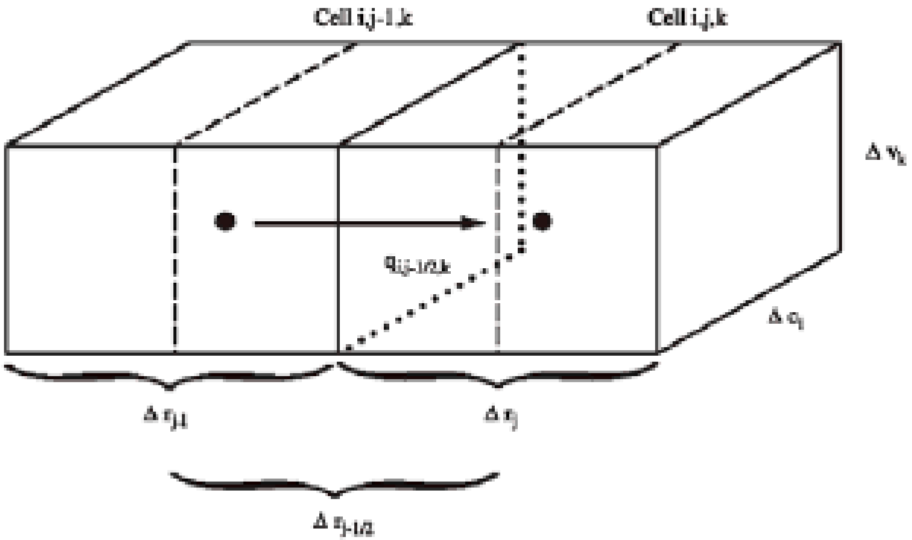
Discretized aquifer showing boundaries an
cell designations. (Modified from McDonal
and Harbaugh, 1988.)



Schemes of vertical discretization.
(From McDonald and Harbaugh, 1988.)

MODFLOW



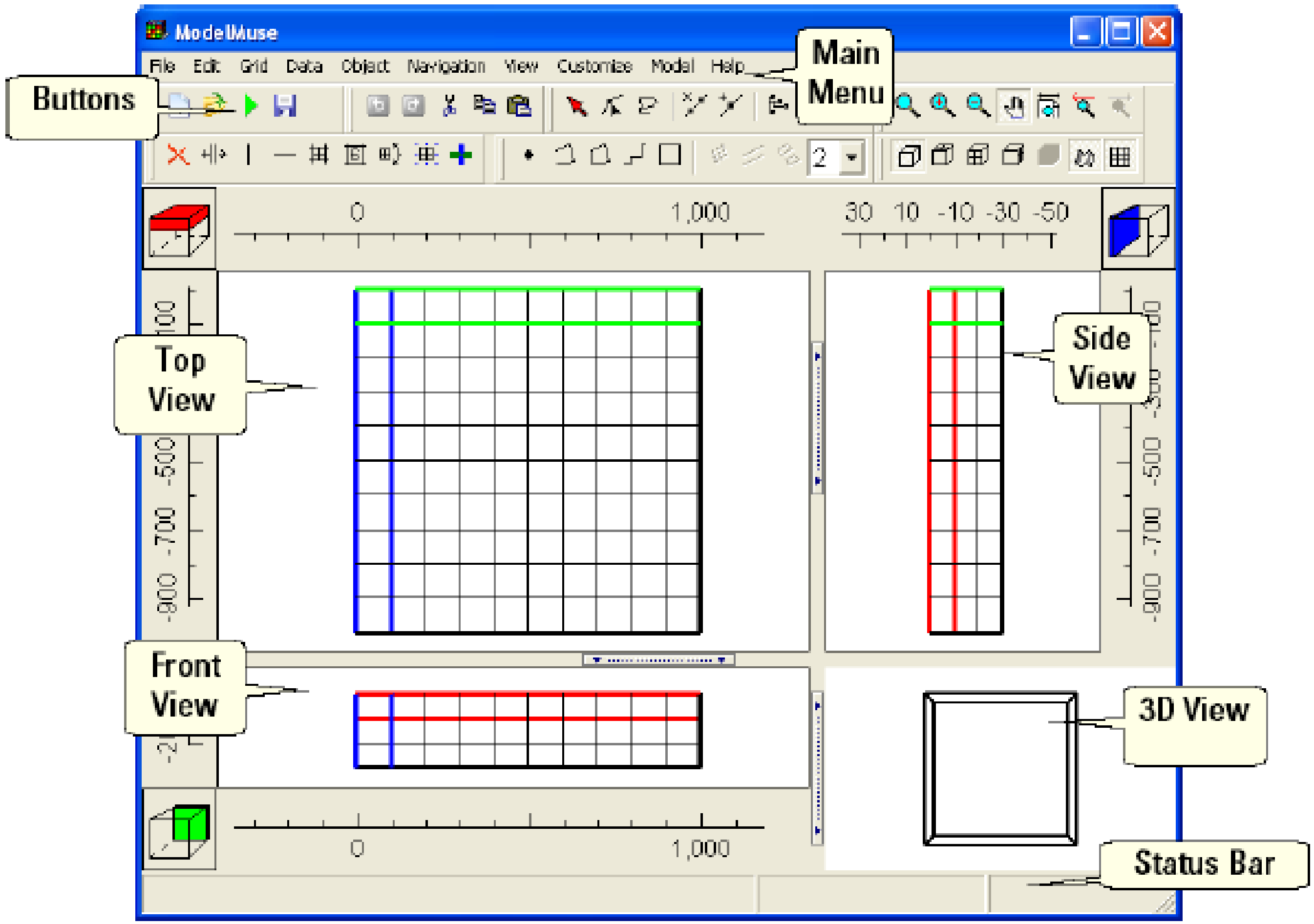


Indicies for the six adjacent cells surrounding cell i,j,k (hidden). (Modified from McDonald and Harbaugh,1988.)

Flow into cell i,j,k from cell i,j-1,k. (Modified from McDonald and Harbaugh, 1988.)

MODELMUSE

- A Graphical User Interface for MODFLOW–2005 and PHAST



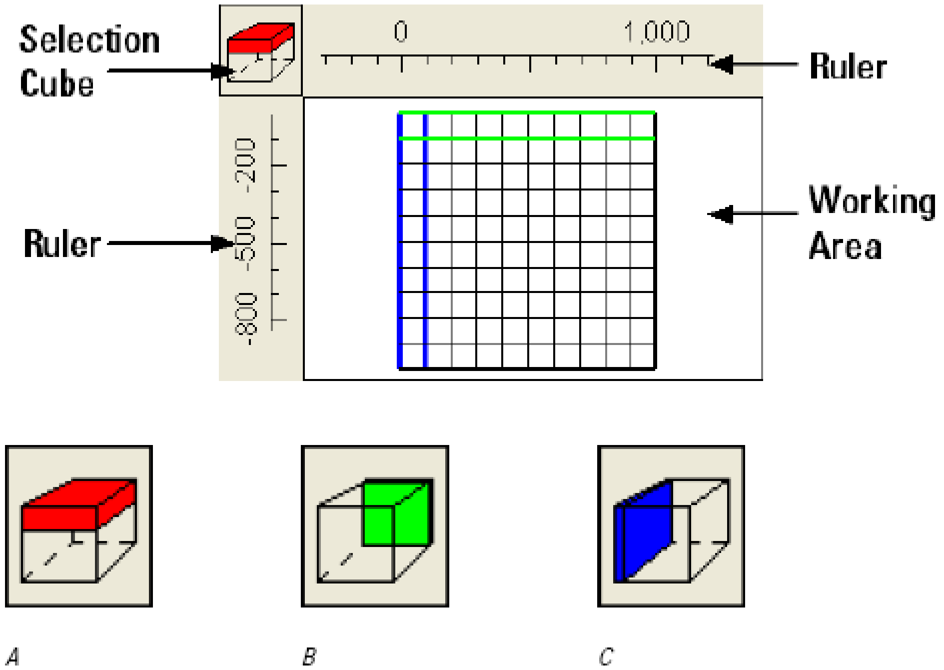
The main window of ModelMuse.

MODELMUSE

Working Area

Top, Front and Side views of the study area domain.

These view options combined with selection of grids will help in assigning the parameters.



- ModelMuse has tools to generate and edit the model grid.
- It also has a variety of interpolation methods and geographic functions that can be used to help define the spatial variability of the model.

MODELMUSE

- Work with any of the three examples
- Run the model and interpret the results
- Investigate the effect of changing some pumping rates and boundary conditions, hydraulic conductivities on the flow regime – head distribution, flows across the boundaries, etc.

Learning Outcomes

If completed correctly, you will be able to

- Define hydraulic head and gradient of hydraulic head
- Apply Darcy's law
- Distinguish the governing equations for the
  - confined and unconfined aquifers;
  - steady and unsteady conditions
- Solve the equations graphically using flownets
- Develop and apply equations to
  - estimate the steady flows in confined and unconfined aquifers
  - Calculate the water table level at different locations in unconfined aquifers
- Modeling Basics and MODFLOW Set up using ModelMuse